THE USE OF HISTORY OF MATHEMATICS IN LEARNING THE EXTRACTION OF SQUARE ROOTS

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Abstract

This study aimed at implementing the instructional activities of learning the extraction of square roots based on Liu Hui's geometrical approach. The mathematical problems were designed based on the instructional sequence in which Realistic Mathematics Education (RME) approach was used as the base for the learning. Design research was chosen to achieve this research aim. In design research, the instructional sequence plays important role as a design and research instrument. It was designed in the phase of preliminary design and tested to three Bachelor level students at Middle East Technical University, Turkey. The designed instructional sequence was then compared with students' actual learning in the experimental phase and analyzed in which students learned or did not learn from what we had conjectured them to learn in the retrospective analysis phase. The result of the experiments showed that the context of finding the length of the fence (for a square land), coupled with geometry approach based on Liu Hui's method, has a potential for learning the extraction of square roots.

Keywords: extraction of square roots, Liu Hui's geometrical approach, Realistic Mathematics Education

1. INTRODUCTION

In primary school, the students are introduced basic mathematics operations (i.e., arithmetic), namely addition, subtraction, multiplication, and division. As argued by Bramall and White (2000), most people realize that they need to know and learn arithmetic as it will be used in daily life. For instance, arithmetic used in routine activities such as finding out time by looking at our watch, comparing prices of shirts we want to buy, deciding on quantities of different items we need to purchase for a party, and so forth. However, when it comes to algebra (e.g., the extraction of square root, quadratic equations, integral, etc), as argued by Usiskin (1995), most students found some difficulties. The value of algebra is not as obvious as the value of arithmetic. Therefore, before the students get into algebra, they must learn to use symbols as a language in which they can express their own ideas. Then algebra will not be just a meaningless collection of rules of algorithm.

Focus on the topic that will be discussed in this paper, frequently, when we ask our students to find the square root of certain number, they will use calculator to find the solution. They often argue that calculator could do the mathematical calculation and it is not necessary to know the algorithm or formula behind the process of finding the solution. Later, when the students are introduced the algorithm of finding square root of a number, they probably do not know how difficult mathematicians in the ancient times gained this algorithm. That is one of the reasons why we need to introduce students how mathematics growth through history and let them to think as ancient people think in the past. Besides, it is also important for students not only knowing the algorithm but also realizing how it works and how it was invented by mathematicians. By understanding how mathematician invented this algorithm, it could give students more opportunity to think and learn mathematics meaningfully.

According to Fauvel & van Maanen (2000), by studying history and trying to reconstruct aspects of the historical development of specific mathematical topics in a didactically appropriate manner, teachers may "identify the motivations behind the introduction of (new) mathematical knowledge, through the study of examples that served as prototypes in its historical development and which may help students to understand it". However, some people may assume that the history of mathematics is something outside from mathematics. For example in ICMI study (Fauvel & van Maanen, *ibid.,* p. 203), there are some arguments against the incorporation of history and mathematics. One of them asserted that history is not mathematics, and if you want to teach history, then firstly, you need to teach mathematics itself. This assumption is not in general but it can be agreed upon. However, through this study, we want to show that integrating history of mathematics in the mathematics learning process is useful not only for students but also for the teacher itself.

There are some algorithms which related to the topic of the extraction of square root gained by mathematicians in the past. For instance, the procedure given by an Aryabhata (499) collected with their Sanskrit transliteration (Keller, 2007), finding the approximate values for the square root of numbers by Ibrahim Hakkı of Erzurum (Tatar, et.al., 2010), and Liu Hui's extraction of the square root (Martzloff, 2006) by using geometry approach. These three historical procedures of extracting square roots will be explained further in the theoretical framework.

In this study, we then decided to choose Liu Hui (c. 250) method in which geometry approach was used to find the solution and to generate the formal algorithm. Besides, we also tried to design an instructional activities built based on Realistic Mathematics Education (RME) developed by Freudenthal (Gravemeijer, 1994). This method is based on the idea of mathematics as a human activity and as a constructive activity. Therefore, to facilitate the learning of extracting square root, we used context of finding the length of fence, a real life situation which is familiar for the students. Besides, a square figure was used as a provoking tool to bring the students to the idea of extracting square roots as repeated subtraction of squares and divisions by a doubled number.

To summarize, the aim of this study was to implement the instructional sequence of learning the extraction of square root. It was expected that through the use of geometrical approach, the students could reinvent the algorithm of extracting square roots. Thus, the research question was formulated as follows.

"How can geometry approach inspired by Liu Hui support the students in extracting square roots?"

2. THEORETICAL FRAMEWORK

2.1. Historical Procedures for Extracting Square Roots.

As mentioned in the introduction, the three historical procedures of extracting square roots from Indian (Aryabhata), Turkish (Ibrahim Hakkı of Erzurum), and Chinese (Liu Hui) mathematicians will be discussed as follows.

2.1.1. Aryabhata's Root Extraction Method

Aryabhata was born in 476. He occupies an important place in history of mathematics and astronomy. Aryabhata's book explicates his astronomical and mathematical theories. The books by Data and Singh and Srinivasiengar in Keller (2007) are good sources for history of Indian computing algorithm. In this subchapter we will report Aryabhata's root extraction methods given in the mathematical section of *Aryabhatiya*, which is called the *Ganitapada*. Keller in her analysis showed that Aryabatha's method was different from that of Greek Mathematics. Aryabatha was well aware of place value system of number. Below is an example of extracting the square roots of a three digit number (table 1).

Table 1. An example of extracting the square root of a three digit number.

Aryabhata's Rule	**Example: Extracting the Square Root of 625**	**Extracting the Square Root of** $A=(a.10+b)^2$
When subtracting the square from the square <place>.	The biggest square smaller than 6, which is the digit in the "highest square place", is 4. So that 2 is the first digit of the square root to be extracted. Below is how the set number may have been set down. *v* *av* *v* 6 2 5 -4 - - 2 2 5	$A-a^2.10^2$ is computed. $a.10$ is the partial square root extracted.
One should divide, constantly, the non-square <place> by twice the square root.	27 is considered to be the "non-square" place. Twice the partial square root is $2\times2=4$ performed in the following division. $\frac{22}{4}=5+\frac{2}{4}$ 5 is the quotient. It is the second digit of the square root to be extracted. The partial square root is, at this point, 25. The remainder of the division of 22 by 5 is set down in the place of the previously written digits. *av* *v* 2 5	b is computed as the quotient of the division of two higher digits by a^2. Then $A-a^2.10^2-2ab10$ is set down. $a.10+b$ is the partial square root extracted.
The quotient is the root in the next place. When subtracting the square from the square.	The quotient is 5. The next place being a square place, one subtracts the square of 5. *av* *v* 2 5 -5^2 0	$A-a^2.10^2-2ab10-b^2$ is computed.
The square root found is	25	$a.10+b$

2.1.2. Ibrahim Hakkı of Erzurum's Perspective of Square Root Calculation

Ibrahim Hakkı of Erzurum, a Turkish, was born in Pasinler (formerly Hasankale) district in Erzurum on May 18, 1703 AD (1115 AH) in the 18th century Ottoman world when scientific studies were scarce. He produced more than seventy works. In the 4th volume of *Marifetname* (Book of Gnosis), his famous book, he compiled his studies in astronomy and mathematics (geometry, numbers, the four operations with numbers, finding an unknown value, square root calculation, etc). Related to the topic discussed in this paper, we will focus on his finding of calculation for finding square root.

In the 4th volume of *Marifetname*, Hakkı describes a way to find the square root number. He argued that is easy to find the square root number if it is a perfect square (Tatar, et.al., 2010). For instance, the square root of 4 is 2, the square root of 9 is 3, the square root of 25 is 5, and so forth. These numbers are perfect square numbers in which the square roots are whole numbers. Hence, it is easy to find the square root of these numbers. Later, Hakkı generates procedure for finding the square root for a number which is not a perfect square and the square root is not a whole number. The procedures are as follows.

1. Calculate the square root of the next lower perfect square and subtract this number from the whose square root to be found.
2. Then write the result as the numerator.
3. Two times the square root of the closest perfect square plus 1 as the denominator.

4. And, the square root of the perfect square number as the whole part to find the square root of number.

For instance, we want to find square root of 7. We subtract 4, the closest perfect square, from 7 ($7-4=3$). We write the quotient as the numerator. We find the square root of 4 ($\sqrt{4}=2$). We write 2 times the result 2 plus 1 as the denominator ($2 \cdot 2+1=5$) and 2, the square root of 4, as the whole part and thus, we find the approximate value for the square root of 7 is $\sqrt{7}=2\frac{3}{5}\cong 2,6$.

After these examples, Ibrahim Hakkı's square root method can be shown in general formula as follows (*note: proof will not be explained in this paper*).

Let be $\forall x, y, x, a, \sqrt{x} \in \square$ and $\sqrt{y}, \sqrt{a} \notin \square$. Then $x<a<y,$ $\sqrt{y} \cong \sqrt{x}\,\frac{y-x}{2.\sqrt{x}+1}$

2.1.3. Liu Hui's Method for Extracting the Square Roots

Liu Hui (c. 250) lived in north-central China who gave commentary on the Jiu Zhang Suan Shu (Fauvel and Maanen, *op.cit.,* p. 220). It is commonly called as Nine Chapters on the Mathematical Art. It is the longest surviving and one of the most important in the ten ancient Chinese mathematical books. The book that he made is broken up into nine chapters containing 246 questions with their solutions and procedures. Each chapter deals with specific field of questions. One of those chapters is about finding the square root or cube root of a number.

Needham (1995) stated that the geometrical basis of the square root method was almost certainly pictured by Liu Hui. Liu Hui made the area that represents the number whose root he wanted to extract in different color. In addition, according to Martzloff (2006), Liu Hui explained that to find the extraction of the square root, he made a square that represented the area of the number A, the root that he wanted to extract (see figure 1) and try to find the unknown side of this area. He dissected the area *A* by dividing it into yellow (*huang*), red (*zhu*) and azure (*qing*) areas.

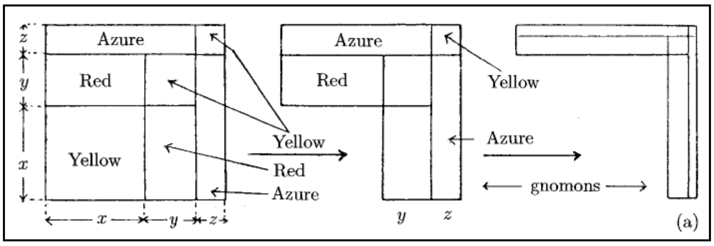

Figure 1. Reconstruction of Liu Hui's figures to justify a technique for extracting the square root (Martzloff 2006, p. 223).

From the figure above, it can be inferred that Liu Hui tried to find the unknown side of the square by finding maximum area related to the square of the number of (for instance) hundreds or thousands in the root. This area was labeled as x^2 and was represented with yellow area. The leaves area is $(A-x^2)$ in the shape of gnomon. To find the remainder area, he defined y, the small side of the rectangle then he called this as "red surfaces" (*zhu mi*), and the large side that equal to the x, side in the previous (yellow) square. According to the figure, the total area of these two rectangles is less than the area of the remainder gnomon. Where:

$$y \leq \frac{(A-x^2)}{2x}$$

This inequality was used to determine the second figure of square root by trial and error, by dividing the two numbers $(A-x^2)$ and $2x$ (both known) by one another (note that Liu Hui called *x* the "fived divisor" *ding fa*).

Next step, he subtracted the two rectangles and the small square area (y^2) from the gnomon. Then he did the same thing to find the area of the remainder of that gnomon.

2.2. Realistic Mathematics Education (RME)

Realistic Mathematics Education (RME) is a theory of mathematics education which has been developed in the Netherlands since 1970s. This theory is strongly influenced by Hans Freudenthal's concept of 'mathematics as a human activity' (Freudenthal, 1991). According to Freudenthal, students should not be

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treated as passive recipients of ready-made mathematics, but rather than education should guide the students towards using opportunities to discover and reinvent mathematics by doing it themselves.

As explained previously, also connected to our research question, Liu Hui's algorithm of extracting square root gives inspiration for us to develop instructional activities in finding the square root of a number without using calculator. In designing the instructional activities, we followed the principle of Realistic Mathematics Education (RME) approach. In general, RME is characterized by several features (Sutherland, 2004), namely:

1. The use of realistic (as in "real to the learner") context as sources from which to develop mathematics and as situations in which the problems to be solved are presented. Therefore, in our instructional activities, we used *finding the length of the fence* context as starting point to guide the students for finding the square root.
2. The use of models to develop mathematical concepts, thinking and skills. This model is designed in order to allow students to move from the concrete informal level of doing mathematics to the more abstract and formal level. This process is known as *progressive formalization*. In our instructional material, we used geometrical approach in which square is used as visualization to find square root of numbers.
3. The use of *guided reinvention*. Students are encouraged to reinvent the mathematics, guided by the teacher or facilitator and the instructional activities. In our instructional activities, we guided and provided students the opportunity to experience a process similar to the process by which mathematics was invented.
4. The use of various instructional modes (individual, group work, pairs, with and without technology), together with interaction (e.g., through discussion) is the key to explicate the learning and teaching. In our instructional material, we used discussion during the learning.
5. The use of intertwined learning strands. Mathematics is seen as one subject. At primary and secondary school, there are no separate courses on algebra, geometry, and calculus, but the topics are integrated in one course names mathematics. Therefore, in our instructional material we tried to integrate the process of finding square root by using geometry approach.

3. RESEARCH METHOD

3.1. Design Research

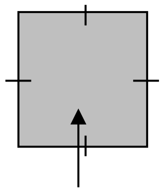
The type of research that we used is design research (Gravemeijer & Van Eerde, 2009) that also termed as developmental research because instructional materials were developed. Design research consists of three phases, namely developing a preliminary design, conducting pilot and teaching experiments, and carrying out a retrospective analysis (Gravemeijer, 1994; Bakker, 2004). Three bachelor degree students from different departments in Middle East Technical University (METU), Turkey, were involved in this research. Table 2 shows the list of participants and their backgrounds (*note: researcher has already gotten permission for mentioning their names in this paper*). The data collected in this research were interviews with the research subjects, field notes, and research subjects' works.

Table 2. List of participants and their backgrounds.

Name	Background
Kemal Jacob Alis	Biology (8^{th} semester)
Romadhona Febriarisqa	Mathematics (2^{nd} semester)
Teuku Muzafarsyah	Computer Education Instructional Technology (6^{th} semester)

The data collected in this research were interviews with the research subjects, field notes, and participants' works. After we collected the data, we analyzed these data in the retrospective analysis. Finally, we made conclusion based on the retrospective analysis. The instructional activities of learning the extraction of square root are as follows (table 3).

Table 3. The instructional activities of learning the extraction of square root.

Learning Goal	Activity	Conjecture of Students' Strategies	Conjecture of Students' Difficulties
Students will make interpretation of the word problem.	Interpreting the word problem.	- Students will connect the problem with "square". - Students will draw the problem geometrically, e.g.,  Area = 16,129 m^2	- Students may have difficulties in translating the problem into mathematical expression.
Students will find the length of a square to find the total length for fencing the land.	Finding the length of the land's side without using calculator and finding the total length for fencing the land.	- Students will try to find the square of 16,129 to find the length of land's side. It is based on the concept of area of a square. - Students will use geometry approach to find the length of the square.	Students cannot connect the problem with the idea of the area of square. Therefore they cannot notice that to find the length of a square is by finding the square root of area of the land.
Students will generate the procedure for extracting square root	Generating procedure for extracting square root	The students will see the pattern from the result of extracting square root. The pattern is repeated subtractions of square and divisions by a doubled number.	Students cannot see the pattern or they use different approach. Therefore they cannot generalize the procedure.

4. RETROSPECTIVE ANALYSIS

4.1. Interpreting the Word Problem

As mentioned in the first tenets of Realistic Mathematics Education (RME), contextual problem figured as application and as starting point from which the intended mathematics could come out. For that reason, the context of *finding the length of the fence* was chosen as the context in which the students could connect their real life experience with the idea of extracting square roots.

In this activity, the students were asked to interpret the problem of *"Mr. Pratama wants to put a fence around 4 sides of his square land that has an area of 16,129 m^2. How meters fencing does he need?"*. It was found that all students have no difficulties in interpreting this problem. They could see the keywords such as "*square land and its area which is 16,129 m^2*". They also knew that to find the total length for fencing the land is by firstly finding the length of square side. Connected to their prior knowledge, about the area of square, all students said *"to find the length of the side, we just need to find the square root of 16,129"*. However, when they wanted to find the length of square side, they were directly asked whether they could use calculator or not. To guide them to Liu Hui's idea to find the extraction of square root, we tried to provoke them to use geometrical approach.

4.2. Finding the length of the land's side without using calculator and finding the total length for fencing the land

As mentioned in the second tenet of RME, namely the use of models to develop mathematical concepts, thinking and skills, in this activity geometrical approach was used. This model was designed in order to allow students to move from the concrete informal level of doing mathematics to the more abstract and formal level. This process is known as *progressive formalization*.

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In this activity, the students were provoked to draw a square as a representation of the square land and noted that the area is 16,129 m^2. Most of them directly used algebraic symbols in their drawings to label the known and unknown as shown in figure 2.

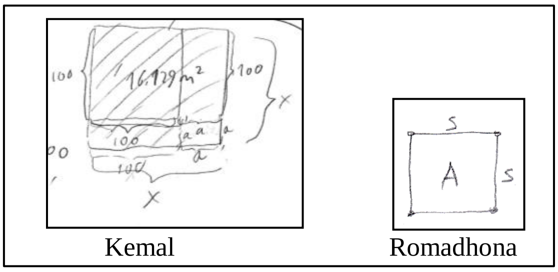

Figure 2. Students' drawing for the representation of square land and its area.

We also tried to provoke the students to use the idea of repeated subtraction of area as proposed by Liu Hui and said to them *"think as students in grade 6 think"* as this topic is actually given to 6 grader students. However, instead of using repeated subtraction of area, most of the students used their bachelor level knowledge about mathematics except Romadhona. Each student's strategy will be explained in detail as follows.

First strategy that will be discussed came from Teuku. Differ from his other friends, he did not draw the representation of square land and decided to use the formula of *tangent line approximation* which can be seen in the figure 3.

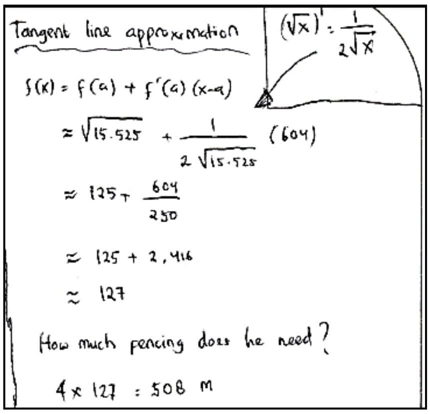

Figure 3. Teuku's strategy in solving problem using tangent line approximation.

As we can see from Teuku's strategy, he directly used formula and found out that $f(a)=\sqrt{15,525}$ and $f'(a)=\frac{1}{2\sqrt{15,525}}(604)$. As we asked him to explain this, he said that it was the formula of tangent line approximation, in which we need to find the square of a number that near to 16,129 which is 15,525. However, Teuku miscalculated the square number near to 16,129 which is supposed to be 15,625. Then, following the procedure, he added $f(a)$ with $f'(a)$. 604 is the result of subtracting 16,129 with 15,525 which is supposed to be 504 from the result of subtracting 16,129 with 15,625. If we connect this tangent line approximation with Ibrahim Hakkı's formula for finding the extraction of square root, this formula is almost similar with Teuku's formula. If we try to solve this problem by using Ibahim Hakkı's formula, the result as follows.

$$\begin{aligned}\sqrt{16,129} &= \sqrt{15,625}\,\frac{16,129-15,625}{2\times\sqrt{15,625}+1}\\ &= 125\,\frac{504}{2\times 125+1}\\ &= 125\,\frac{504}{251}\\ &\cong 127.008\end{aligned}$$

Even Teuku did miscalculation in determining a square number which is near to 16,129, his final answer was correct since he approximated $125+\frac{604}{250}\approx 125+2,416\approx 127$.

Another strategy was used by Kemal as shown in the figure 4 below together with his explanation.

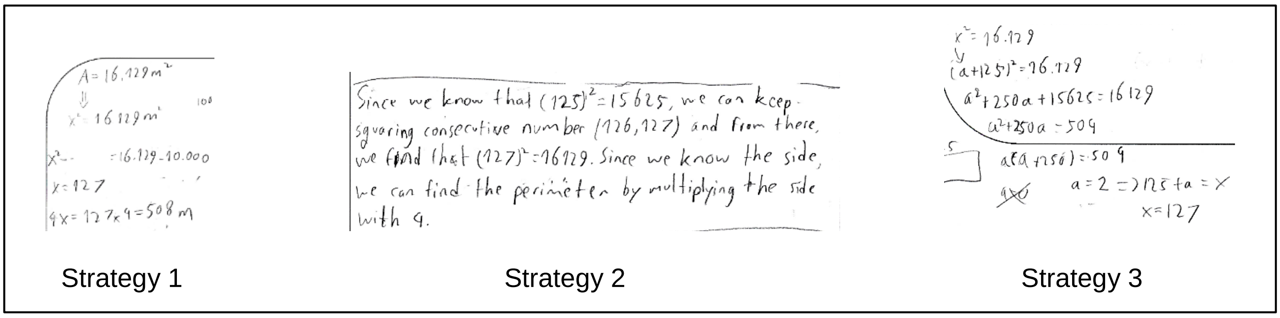

Figure 4. Kemal's strategies for finding square roots number.

Kemal came up with 3 strategies in solving the problem which we did not conjecture in our instructional activities. However, his ideas can enrich our instructional activity in the part of conjecture of students' thinking. In his first strategy, we could not grasp his idea since there is no further calculation after his calculation $x^2 - \_\_\_\_ = 16,129 - 10,000$ but then suddenly he came up with the solution that $x = 127$. In order to clarify his solution, we asked him to explain his answer. However, he used different strategy (strategy 2) to explain his first strategy. He used the idea of finding square number which close to 16,129, namely 15,625 and the root is 125. Then, by squaring the consecutive number, he found that the length of square side is 127. He then admitted that he used trial and error for his second strategy.

For his third strategy, we may interpret that he used the idea of finding the factorization of quadratic equation. But first, he decided that *x*, the length of square side was equal to $a+125$. Then, he just needed to find the rest length of square side defined as *a*. By using quadratic equation procedure, Kemal could find the length of square side, namely 127 as we can see from figure 4.

The strategy in line with our conjectures came from Romadhona's strategy. He used repeated subtraction of area to extract the square root number. Romadhona's strategy was quiet clear since he also provided the steps geometrically as shown in figure 5 below.

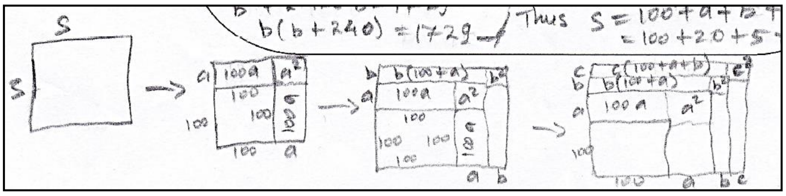

Figure 5. Romadhona's steps represented geometrically.

From Romadhona's drawing, we can see that Romadhona defined the total length of the side as 100 + *a* + *b* + *c*. However, in finding the solution, Romadhona used formal calculation as shown in figure 6 below.

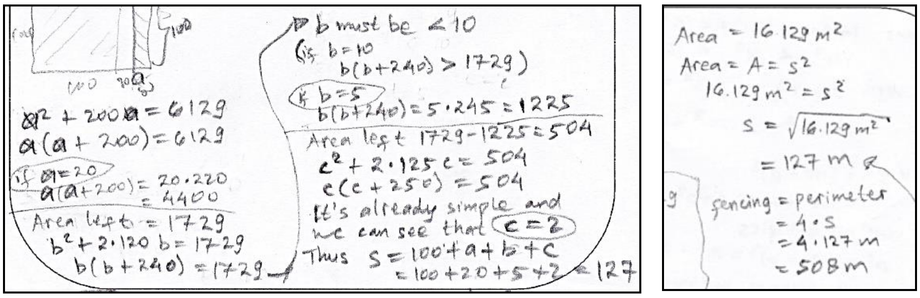

Figure 6. Romadhona's algebraic procedures.

As we can see from Romadhona's idea, the idea of repeated subtraction of area can be seen from his notes, namely *"area left"*. He also used factorization, a more formal strategy, to find the length of *a*, *b*, and *c*. Besides, he also used approximation as we can see from his note: "b must be <10". After doing repeated subtraction of area, then he summed up 100 + *a* + *b* + *c* which is equal to 127. Then, he finally found the total length for fencing the square land as $4 \cdot s = 4.127 = 508$ meters.

Later, we tried to guide Romadhona to see pattern in his solution. It took sometimes for him to finally realize the pattern. Romadhona's struggle in doing repeated subtraction of area and his struggle in seeing the pattern in order to generalize the procedures could be interpreted as difficulties which probably faced by ancient mathematicians. Provoking idea to use geometrical approach and repeated subtraction of area had led him to the insight of extracting square roots without using calculator.

4.3. Generating procedure for extracting square roots

In this activity, since Teuku and Kemal used formula (tangent line approximation), trial and error, and approximation strategies, they could not see any pattern from their solutions. Therefore, we asked Romadhona to generalize the pattern of his solution. Below is his explanation in generalizing the procedures (figure 7).

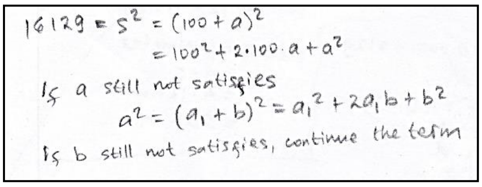

Figure 7. Romadhona's generalization procedures for extracting square roots seeing from the pattern in his solution.

From his notes, Romadhona found out the pattern of square numbers from the repeated subtraction of area. He also noted that if *"still not satisfies, continue the term"*. This indicates the repetition procedure, namely repeated subtraction of area. Besides, Romadhona also provided another generalization procedure for extracting square roots as shown in figure 8 below.

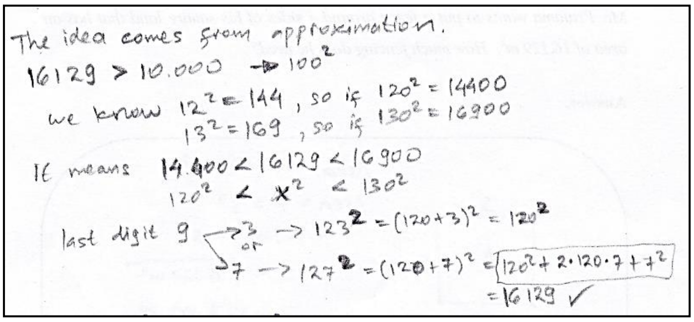

Figure 8.Romadhona's generalization procedures for extracting square root based on the idea of approximation.

From the figure above, it can be seen that his second generalization was not based on pattern in his solution. He used the idea of approximation and it is also related to trial and error strategy as used by Kemal. However, Romadhona's generalization from his solution pattern has contributed to the learning even though it has not reached what we have conjectured that the students could reinvent the algorithm for extracting square roots. Therefore, such an additional activity which could guide the students to reinvent the algorithm for extracting square roots should be designed in order to reach the learning goal.

5. Conclusion and Recommendation

Our goal in this study was to implement the instructional sequence of learning the extraction of square roots based on Liu Hui's geometrical approach. We drew on the work of three mathematicians namely Aryabhata (Keller, 2007), Ibrahim Hakkı of Erzurum (Tatar, et.al., 2010), and Liu Hui (Martzloff, 2006) to find the way they extracted square roots.

Our findings suggested that grounding students' work in the context of *finding the length of the fence (for a square land)*, coupled with geometry approach, has a potential for learning the extraction of square roots.

The mathematical practices found that three students, the participant in this study, who are now in the university level, struggled to solve the problem involving extracting square root. They used their prior university level knowledge to solve the problem as done by Teuku when he used *tangent line approximation* to find the square root number. Moreover, the students were also prefer to use approximation or estimation strategy and trial and error as used by Kemal and by Romadhona when he generalized the procedure of extracting square roots. This estimation strategy is important according to Dowker (1992) which argued that many studies underline the importance of the ability to estimate in mathematics and this skill should be improved. In this study, this skill was used by these three students to find the approximate values for the square root of numbers.

To answer the research question, this study has shown that geometry approach can support the students in extracting square roots without using calculator. However, only one out of three students succeeded in reaching this goal. Such limitation occurred in this study in which the implementation should be held in the real classroom and work with 6 graders students. Students' university level knowledge indeed influenced students' views in approaching the problem. Therefore, it is suggested to conduct this research in the real classroom experiment for the next research. Besides, such an additional activity in the instructional activities should be added in order to reach the goal of reinventing the algorithm for extracting square root.

From this study, related to the algebraic thinking, the students have followed the "functional thinking" as proposed by Ministry of Education in Japan (1999), namely (1) identify some independent variables and dependent variable in the problem situation (i.e., when the students tried to interpret the problem and find unknown and known variables), (2) find a recurrent relation of functional relation (i.e., it can be seen when Romadhona decided to use the idea of repeated subtraction of area to extract square root), and (3) use the pattern or relations to solve the problem.

Although the study reported here is a relatively small scale study and only worked with three students, not all findings of which can be generalized, this study has some implications for educational practice. There are also, of course, some possibilities to conduct research in learning the extraction of square root based on other mathematicians.

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